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Bi-conformal vector fields and the local geometric characterization of conformally separable pseudo-Riemannian manifolds I

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Abstract

This is the first of two companion papers in which a thorough study of the normal form and the first integrability conditions arising from *bi-conformal vector fields* is presented. These new symmetry transformations were introduced in *Class. Quantum Grav.* 21, 2153–2177 and some of their basic properties were addressed there. Bi-conformal vector fields are defined on a pseudo-Riemannian manifold V through the differential conditions $\pounds_{\vec{\xi}}\, P_{ab} = \phi P_{ab}$ and $\pounds_{\vec{\xi}}\, \Pi_{ab} = \chi \Pi_{ab}$ where P_{ab} and Π_{ab} are orthogonal and complementary projectors with respect to the metric tensor g_{ab}. In our calculations a new affine connection (*bi-conformal connection*) arises quite naturally and this connection enables us to find a local characterization of *conformally separable* pseudo-Riemannian manifolds (also called double twisted products) in terms of the vanishing of a rank three tensor T_{abc}. Similar local characterizations are found for the most important particular cases such as (double) warped products, twisted products and conformally reducible spaces.

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1. Introduction

The research of symmetry transformations in Differential Geometry and General Relativity has been an important subject during the years. Here by symmetries we mean a group of transformations of a given pseudo-Riemannian manifold complying with certain geometric property. By far the most studied symmetries are isometries and conformal transformations which are defined through the conditions

$$\pounds_{\vec{\xi}}\, \mathrm{g}_{ab} = 0, \qquad \pounds_{\vec{\xi}}\, \mathrm{g}_{ab} = 2\phi \mathrm{g}_{ab}, \tag{1}$$

where g_{ab} is the metric tensor of the manifold, $\vec{\xi}$ is the *infinitesimal generator* of the transformation and ϕ is a function which we will call *gauge* of the symmetry (this terminology was first employed in [5] and it will be explained later). Infinitesimal generators of these symmetries are known as Killing vectors and conformal Killing vectors, respectively. As is very simple to check they are a Lie algebra with respect to the Lie bracket of vector fields and the transformations generated by these vector fields give rise to subgroups of the diffeomorphism group.

Important questions are the possible dimensions of these Lie algebras and the geometric characterizations of spaces admitting the symmetry. The general answer to these questions can in principle be obtained by solving the differential conditions written above although for general enough cases the explicit evaluation of such solutions gets too complex and other methods are required. Notwithstanding these difficulties, we can obtain easily from the differential conditions the cases in which the Lie algebras are finite dimensional, the greatest dimension of these Lie algebras and geometric characterizations of the spaces admitting these Lie algebras as solutions. This is done by finding the *normal form* of the above equations (if such form exists) and the complete integrability conditions coming from this set of equations. In this way we deduce that isometries are always finite dimensional whereas conformal motions are finite dimensional iff the space dimension is greater or equal than three. The spaces in which the greatest dimension is achieved are constant curvature and conformally flat spaces, respectively and as is very well known they are characterized by the geometric conditions

$$R^a{}_{bcd} = \frac{R}{n(n-1)}(\delta^a{}_c \mathrm{g}_{bd} - \delta^a{}_d \mathrm{g}_{bc}) \text{ (constant curvature)},$$

$$C^a{}_{bcd} = 0,\ n > 3 \text{ (conformally flat)}$$

where n is the dimension of the manifold, $R^a{}_{bcd}$ is the curvature tensor, R the scalar curvature and $C^a{}_{bcd}$ the Weyl tensor[1]

The procedure followed for isometries and conformal motions is carried over to other symmetries such as linear and affine collineations and conformal collineations (see [24,12] for a very good account of this). However, little research has been done for symmetries different from these mostly because the cases under consideration were infinite dimensional *generically*. This means that it is not possible to obtain a normal set of equations out of the

[1] In the case of dimension three manifolds the Weyl tensor is replaced by a three-rank tensor called the Cotton-York (or Schouten) tensor.

differential conditions (see Section 4) which greatly complicates matters. Mathematicians have developed an alternative view toward this issue in the theory of *G-structures* (see e.g. [17] for a thorough description of this).

In reference [10] we put forward a new symmetry transformation for general pseudo-Riemannian manifolds. Infinitesimal generators of these symmetries (bi-conformal vector fields) fulfill the differential conditions

$$\pounds_{\vec{\xi}}\, P_{ab} = \phi P_{ab}, \qquad \pounds_{\vec{\xi}}\, \Pi_{ab} = \chi \Pi_{ab}, \tag{2}$$

where P_{ab} and Π_{ab} are orthogonal and complementary projectors with respect to the metric tensor g_{ab} and ϕ, χ are the gauges of the symmetry. These are functions which, as happened in the conformal case, depend on the vector field $\vec{\xi}$ so a solution of (2) is formed by $\vec{\xi}$ itself and the gauges ϕ and χ (we will usually omit the dependence on $\vec{\xi}$ in the gauges). The finite transformations generated by bi-conformal vector fields are called bi-conformal transformations. In a sense, these symmetries can be regarded as conformal transformations with respect to both P_{ab} and Π_{ab} so we can expect that some properties of bi-conformal vector fields will resemble those of conformal transformations. In [10] it was shown that bi-conformal vector fields comprise a Lie algebra under the Lie bracket and that this algebra is finite dimensional if none of the projectors has algebraic rank one or two being the greatest dimension

$$N = \tfrac{1}{2} p(p+1) + \tfrac{1}{2}(n-p)(n-p+1),$$

with p the algebraic rank of one of the projectors. We provided also explicit examples in which this dimension is achieved, namely bi-conformally flat spaces which in local coordinates $x = \{x^1, \dots, x^n\}$ look like $(\alpha, \beta = 1, \dots, p; A, B = p+1, \dots, n)$

$$\mathrm{d}s^2 = \Xi_1(x)\eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \Xi_2(x)\eta_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B, \quad \Xi_1, \Xi_2 \in C^3, \tag{3}$$

where $\eta_{\alpha\beta}$, η_{AB} are flat metrics depending only on the coordinates x^α and x^A respectively. That these spaces play the same role for bi-conformal vector fields as conformally flat spaces or spaces of constant curvature do for the classical symmetries will be a result of the analysis started in this paper. One can also find a geometric characterization for bi-conformally flat spaces similar to those of the spaces of constant curvature or conformally flat spaces stated in (2) (full details of this are contained in [11]). In the scheme developed in [10] this sort of characterization could not be extracted due to the complexity of the calculations and it had to be postponed.

In this paper we perform the full calculation of the normal form for Eq. (2). This normal form is already present in our previous work but it turned out to be rather messy and relevant geometric information could not be obtained. This was so because all these calculations were done using the covariant derivatives arising from the metric connection which is not adapted to the calculations. Here we show that the definition of a new symmetric connection (*bi-conformal connection*) greatly simplifies the calculations making it possible to get a simpler form for the normal system. Due to the great amount of algebra required to work out the complete integrability conditions associated to the normal form we have placed its analysis together with the geometric characterization of the maximal spaces in a subsequent paper (a complete version of all our results can be found in [11]).

The bi-conformal connection bears an interesting geometric interpretation if we work on *conformally separable* pseudo-Riemannian manifolds. These are defined as those manifolds which in a local coordinate system the metric tensor takes the form (the conventions are the same as in (3))

$$\mathrm{d}s^2 = \Xi_1(x) g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \Xi_2(x) G_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B,$$

where $g_{\alpha\beta}$ and G_{AB} only depend on the coordinates labelled by their index components. The bi-conformal connection is naturally adapted to this decomposition and its use permits us to give a new simple geometric characterization of these spaces in terms of the vanishing of a certain rank-three tensor T_{abc}. The most known cases of conformally separable pseudo-Riemannian manifolds are warped products, double warped products, twisted products and conformally reducible spaces (see Definition 12 for a precise account of each case) and we can easily derive with our techniques local geometric characterizations for these spaces (Theorem 16).

The outline of the paper is as follows: Section 2 introduces the basic notation and definitions. In Section 3 we define a new symmetric connection (bi-conformal connection) and we set its main properties. Section 4 presents the calculation of the normal form associated to (2) and the calculation of the maximum dimension of any finite dimensional Lie algebra of bi-conformal vector fields is carried out. In Section 5 we use the bi-conformal connection to supply a local geometric characterization of conformally separable pseudo-Riemannian manifolds and their principal subcases. Finally in Section 6 we show in explicit examples how to use this geometric characterization and we hint how these conditions may be extended to more general pseudo-Riemannian manifolds. Appendix A collects basic identities relating the Lie derivative and the covariant derivative.

2. Bi-conformal vector fields and bi-conformal transformations

Let us start by setting our notation and conventions for the paper. We work on a differentiable manifold V in which a C^∞ metric g_{ab} of arbitrary signature has been defined (pseudo-Riemannian manifold). Vectors and vector fields are denoted with arrowed characters $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$ (we leave to the context the distinction between each of these entities) when expressed in coordinate-free notation whereas 1-forms are written in bold characters $\boldsymbol{u}$. Sometimes this same notation will be employed for other higher rank objects such as contravariant and covariant tensors. Indexes of tensors are represented by lowercase Latin characters a, $b, \ldots$ and the metric g_{ab} or its inverse g^{ab} are used to respectively raise or lower indexes. Rounded and square brackets are used for symmetrization and antisymmetrization, respectively and whenever a group of indexes is enclosed between strokes they are excluded from the symmetrization or antisymmetrization operation. Partial derivatives with respect to local coordinates are $\partial_a \equiv \partial/\partial x^a$. The Levi-Civita connection associated to g_{ab} is $\gamma^a{}_{bc}$ (Ricci rotation coefficients) reserving the symbol $\Gamma^a{}_{bc}$ only for the Christoffel symbols, namely, the connection components in a natural basis. The covariant derivative and the Riemann tensor constructed from this connection are denoted by ∇ and $R^a{}_{bcd}$ respectively being our convention for the Riemann tensor

$$R^a{}_{bcd} \equiv \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{rc}\Gamma^r{}_{db} - \Gamma^a{}_{rd}\Gamma^r{}_{cb}. \tag{4}$$

Under this convention the Ricci identity becomes

$$\nabla_b\nabla_c u^a - \nabla_c\nabla_b u^a = R^a{}_{rbc}u^r, \qquad \nabla_b\nabla_c u_a - \nabla_c\nabla_b u_a = -R^r{}_{abc}u_r.$$

All the above relations are still valid for a non-metric symmetric connection.

The set of smooth vector fields of the manifold V is denoted by $\mathfrak{X}(V)$. This is an infinite dimensional Lie algebra which is sometimes regarded as the Lie algebra of the group of diffeomorphisms of the manifold V. Finally the Lie derivative operator with respect to a vector field $\vec{\xi}$ is $\pounds_{\vec{\xi}}$.

One of the main subjects of this paper is the study of bi-conformal vector fields whose definition given in [10] we reproduce here.

Definition 1. A smooth vector field $\vec{\xi}$ on V is said to be a *bi-conformal vector field* if it fulfills the condition

$$\pounds_{\vec{\xi}}\, P_{ab} = \phi P_{ab}, \qquad \pounds_{\vec{\xi}}\, \Pi_{ab} = \chi \Pi_{ab}, \tag{5}$$

for some functions $\phi, \chi \in C^\infty(V)$.

P_{ab} and Π_{ab} are smooth sections of the tensor bundle $T^0_2(V)$ such that at each point $x \in V$ they form a pair of orthogonal and complementary projectors with respect to the metric tensor $g_{ab}|_x$. This leads to

$$P_{ab} = P_{ba}, \qquad \Pi_{ab} = \Pi_{ba}, \qquad P_{ab} + \Pi_{ab} = g_{ab}, \qquad P_{ap}P^p{}_b = P_{ab},$$
$$\Pi_{ap}\Pi^p{}_b = \Pi_{ab}, \qquad P_{ap}\Pi^p{}_b = 0.$$

Eq. (5) can be re-written in a number of equivalent ways as shown next. To that end we define the tensor S_{ab} in terms of the projectors P_{ab} and Π_{ab} by

$$S_{ab} \equiv P_{ab} - \Pi_{ab} \Rightarrow P_{ab} = \tfrac{1}{2}(g_{ab} + S_{ab}),$$
$$\Pi_{ab} = \tfrac{1}{2}(g_{ab} - S_{ab}) \Rightarrow S_{ap}S^p{}_b = g_{ab}. \tag{6}$$

The last property of this set means that S_{ab} is a square root of the metric tensor. It is not difficult to prove that the endomorphism $S^a{}_b$ can be always diagonalized and the only possible eigenvalues are $+1$ and -1 being the associated eigenspaces the subspaces upon which the projectors $P^a{}_b$ and $\Pi^a{}_b$ project. Other interesting point is that a square root is always the *superenergy* of a simple form (see [10] for further details).

In terms of the square root S_{ab} the conditions (5) take the equivalent form

$$\pounds_{\vec{\xi}}\, g_{ab} = \alpha g_{ab} + \beta S_{ab}, \qquad \pounds_{\vec{\xi}}\, S_{ab} = \alpha S_{ab} + \beta g_{ab},$$
$$\alpha = \tfrac{1}{2}(\phi + \chi), \qquad \beta = \tfrac{1}{2}(\phi - \chi). \tag{7}$$

From Eq. (6) we deduce that both projectors are fixed by the square root S_{ab} so we can use the latter instead of the projectors when working with a given set of bi-conformal vector fields. Following [10] the set of bi-conformal vector fields possessing S_{ab} as the associated square root will be denoted by $\mathcal{G}(S)$. In this paper only expressions involving P_{ab} and Π_{ab} will be used in our calculations. A very important property of $\mathcal{G}(S)$ is that it forms a Lie subalgebra of $\mathfrak{X}(V)$ (proposition 5.2 of [10]) which can be finite or infinite dimensional. Conditions upon the tensor S_{ab} (or equivalently the projectors) for this Lie algebra to be finite dimensional were given in [10] and they will be re-derived in Section 4 in a more efficient way. Observe that the functions ϕ and χ appearing in Definition 1 (or α and β) do depend on the bi-conformal vector field $\vec{\xi}$ (this dependence can be dropped if we work with a single bi-conformal vector field but it should be added when working with Lie algebras of bi-conformal vector fields). In the latter case ϕ and χ (α and β) are called *gauge functions* (see [5] for an explanation of this terminology).

The next set of relations comes straight away from (5)

$$\pounds_{\vec{\xi}}\, P^a{}_b = \pounds_{\vec{\xi}}\, \Pi^a{}_b = 0, \qquad \pounds_{\vec{\xi}}\, P^{ab} = -\phi P^{ab}, \qquad \pounds_{\vec{\xi}}\, \Pi^{ab} = -\chi \Pi^{ab}. \tag{8}$$

Here the last pair of equations are equivalent to (5).

3. The bi-conformal connection

As we have commented in the introduction the Levi-Civita connection $\gamma^a{}_{bc}$ is not suitable to study the normal form and the integrability conditions coming from the differential condition (5) as they result in rather cumbersome expressions. In order to proceed further in our study we are going to show next that the definition of a new symmetric connection greatly simplifies the normal form calculated in [10] and what is more, it will enable us to work out thoroughly the complete integrability conditions arising from this normal form in a subsequent work.

To start with we recall some identities satisfied by any bi-conformal vector field $\vec{\xi}$ which were obtained in [10]. These identities are in fact linear combinations of the first covariant derivative of (5) and we also indicate briefly how they are obtained as this information will be needed later. Using Eq. (A.4) we easily obtain the Lie derivative of the metric connection $\gamma^a{}_{bc}$ ($\phi_b \equiv \partial_b \phi$, $\chi_b \equiv \partial_b \chi$)

$$\pounds_{\vec{\xi}}\, \gamma^a{}_{bc} = \tfrac{1}{2}(\phi_b P^a{}_c + \phi_c P^a{}_b - \phi^a P_{bc} + \chi_b \Pi^a{}_c + \chi_c \Pi^a{}_b - \chi^a \Pi_{cb} + (\phi - \chi) M^a{}_{bc}), \tag{9}$$

where the tensor M_{abc} is defined by

$$M_{abc} \equiv \nabla_b P_{ac} + \nabla_c P_{ab} - \nabla_a P_{bc}. \tag{10}$$

The Lie derivative of M_{abc} can be worked out by means of (A.2) getting

$$\begin{aligned}\pounds_{\vec{\xi}}\, M_{abc} &= \phi M_{abc} + (\chi - \phi) P_{ap} M^p{}_{bc} - P_{bc} \Pi_{ap} \phi^p + \Pi_{cb} P_{ap} \chi^p \\ &= \chi M_{abc} + (\phi - \chi) \Pi_{ap} M^p{}_{bc} - P_{bc} \Pi_{ap} \phi^p + \Pi_{cb} P_{ap} \chi^p,\end{aligned} \tag{11}$$

from which, projecting down with P^{bc} and Π^{bc}, we deduce

$$\pounds_{\vec{\xi}}\, E_a = -p\Pi_{ap}\phi^p, \qquad \pounds_{\vec{\xi}}\, W_a = (p-n)P_{ap}\chi^p, \tag{12}$$

with the definitions

$$E_a \equiv M_{acb}P^{cb}, \qquad W_a \equiv -M_{acb}\Pi^{cb}, \qquad p = P^a{}_a.$$

The following algebraic properties of the tensors E_a and W_a are useful

$$\Pi_{ac}E^c = E_a, \qquad P_{ac}W^c = W_a, \qquad 0 = P^{ab}E_b = \Pi^{ab}W_b. \tag{13}$$

Now, we substitute (12) into (11) yielding

$$\pounds_{\vec{\xi}} \left(M_{acb} - \frac{1}{p}E_aP_{bc} + \frac{1}{n-p}W_a\Pi_{cb} \right) = \phi \left(\Pi_{ap}M^p{}_{cb} - \frac{E_aP_{bc}}{p} \right) + \chi \left(P_{ap}M^p{}_{bc} + \frac{\Pi_{cb}W_a}{n-p} \right). \tag{14}$$

This equation can be written in a more compact form

$$\pounds_{\vec{\xi}}\, T_{abc} = (\phi\Pi_{ar} + \chi P_{ar})T^r{}_{bc} = \phi B_{abc} + \chi A_{abc}, \tag{15}$$

where the definitions of the tensors T_{abc}, A_{abc}, B_{abc} are

$$T_{abc} \equiv M_{abc} + \frac{1}{n-p}W_a\Pi_{bc} - \frac{1}{p}E_aP_{bc}, \tag{16}$$

$$A_{abc} \equiv P_a{}^dT_{dbc} = P_a{}^dM_{dcb} + \frac{1}{n-p}W_a\Pi_{cb}, \tag{17}$$

$$B_{abc} \equiv \Pi_a{}^dT_{dbc} = \Pi_a{}^dM_{dcb} - \frac{1}{p}E_aP_{cb}. \tag{18}$$

Using Eq. (15) we can calculate the Lie derivatives of $A^a{}_{bc}$ and $B^a{}_{bc}$

$$\pounds_{\vec{\xi}}\, A^a{}_{bc} = (\chi-\phi)A^a{}_{bc}, \qquad \pounds_{\vec{\xi}}\, B^a{}_{bc} = (\phi-\chi)B^a{}_{bc}, \tag{19}$$

a relation which shall be used later.

Let us now use all this information to write the Lie derivative of the connection in a convenient way. Note that in Eq. (11) ϕ_a and χ_a appear projected with $P^a{}_b$ and $\Pi^a{}_b$ respectively suggesting that it could be interesting to write any derivative of ϕ and χ decomposed in transverse and longitudinal parts

$$\phi^*_a \equiv \Pi_{ab}\phi^b, \qquad \bar{\phi}_a \equiv P_{ab}\phi^b, \qquad \chi^*_a \equiv P_{ab}\chi^b, \qquad \bar{\chi}_a \equiv \Pi_{ab}\chi^b. \tag{20}$$

If we perform this decomposition in Eq. (9) and replace the terms ϕ^*_a and χ^*_a by means of (12) we get the relation

$$\pounds_{\vec{\xi}} \left(\gamma^a{}_{bc} + \frac{1}{2p}(E_bP^a{}_c + E_cP^a{}_b - P_{bc}E^a) + \frac{1}{2(n-p)}(W_b\Pi^a{}_c + W_c\Pi^a{}_b - W^a\Pi_{cb}) \right) = \frac{1}{2}(\bar{\phi}_bP^a{}_c + \bar{\phi}_cP^a{}_b - \bar{\phi}^aP_{bc} + \bar{\chi}_b\Pi^a{}_c + \bar{\chi}_c\Pi^a{}_b - \bar{\chi}^a\Pi_{cb}) + \frac{1}{2}(\phi-\chi)T^a{}_{bc}, \tag{21}$$

but from (19), (17) and (18) we easily deduce

$$\pounds_{\vec{\xi}}\,(A^a{}_{bc} - B^a{}_{bc}) = (\chi - \phi)T^a{}_{bc},$$

hence Eq. (21) becomes, after some simplifications

$$2\,\pounds_{\vec{\xi}}\left(\gamma^a{}_{bc} + \frac{1}{2p}(E_b P^a{}_c + E_c P^a{}_b) + \frac{1}{2(n-p)}(W_b \Pi^a{}_c + W_c \Pi^a{}_b) + \frac{1}{2}(P^a{}_p - \Pi^a{}_p)M^p{}_{cb}\right)$$
$$= \bar{\phi}_b P^a{}_c + \bar{\phi}_c P^a{}_b - \bar{\phi}^a P_{cb} + \bar{\chi}_b \Pi^a{}_c + \bar{\chi}_c \Pi^a{}_b - \bar{\chi}^a \Pi_{cb}. \quad (22)$$

The geometric object inside the Lie derivative, denoted by $\bar{\gamma}^a{}_{bc}$, is the sum of the metric connection $\gamma^a{}_{bc}$ plus the rank-3 tensor

$$L^a{}_{bc} \equiv \frac{1}{2p}(E_b P^a{}_c + E_c P^a{}_b) + \frac{1}{2(n-p)}(W_b \Pi^a{}_c + W_c \Pi^a{}_b) + \frac{1}{2}(P^a{}_p - \Pi^a{}_p)M^p{}_{bc}, \quad (23)$$

so it is clear that it represents a new linear connection. As we will see during our calculations this linear connection is fully adapted to the calculations involving bi-conformal vector fields and it will be extensively used in this paper.

Definition 2 (Bi-conformal connection). The connection whose components are given by $\bar{\gamma}^a{}_{bc}$ is called bi-conformal connection. The covariant derivative constructed from the bi-conformal connection shall be denoted by $\bar{\nabla}$ and the curvature tensor constructed from it by $\bar{R}^a{}_{bcd}$.

Since $L^a{}_{bc}$ is symmetric in the indexes bc, we see that the bi-conformal connection is symmetric so it has no torsion and all the identities involving only the covariant derivative $\bar{\nabla}$ or the curvature $\bar{R}^a{}_{bcd}$ remain the same as for the case of a metric connection. However, this connection does not in general stem from a metric tensor as can be seen in explicit examples. This means that certain properties of the curvature tensor of a metric connection are not true for $\bar{R}^a{}_{bcd}$. We recall that for a symmetric connection the Riemann tensor is only antisymmetric in the last pair of indexes.

Connections defined in terms of P_{ab}, Π_{ab} and their covariant derivatives as in (23) have been already considered in the literature in relation with the study of integrable and *parallel* distributions [19–22] (see also [17]). A distribution can be univocally fixed by means of two orthogonal and complementary projectors $P^a{}_b$, $\Pi^a{}_b$ and so the study of these projectors can give us information about the geometric properties of the distribution.

It is not very difficult to derive an identity relating the curvature tensor calculated from the bi-conformal connection and the curvature tensor associated to the connection $\gamma^a{}_{bc}$

$$\bar{R}^a{}_{bcd} = R^a{}_{bcd} + 2\nabla_{[c} L^a{}_{d]b} + 2L^a{}_{r[c} L^r{}_{d]b}, \quad (24)$$

this being a thoroughly general identity for two symmetric connections $\bar{\gamma}^a{}_{bc}$ and $\gamma^a{}_{bc}$ differing in a tensor $L^a{}_{bc}$ ([18], p. 141).

The relation between the covariant derivatives $\bar{\nabla}$ and ∇ acting on any tensor $X^{a_1\ldots a_p}_{\ \ b_1\ldots b_q}$ is

$$\bar{\nabla}_a X^{a_1\ldots a_r}_{\ \ b_1\ldots b_q} = \nabla_a X^{a_1\ldots a_r}_{\ \ b_1\ldots b_q} + \sum_{s=1}^{r} L^{a_s}{}_{ac} X^{a_1\ldots a_{s-1}ca_{s+1}\ldots a_r}_{\ \ b_1\ldots b_q} - \sum_{s=1}^{q} L^c{}_{ab_s} X^{a_1\ldots a_r}_{\ \ b_1\ldots b_{s-1}cb_{s+1}\ldots b_q}, \tag{25}$$

which again has general validity for two symmetric connections whose difference is a tensor $L^a{}_{bc}$ [7]. As a first application of this identity we may compare the covariant derivatives of the tensor $L^a{}_{bc}$ which leads us to the identity

$$\bar{\nabla}_{[a} L^b{}_{c]d} = \nabla_{[a} L^b{}_{c]d} + 2L^b{}_{r[a} L^r{}_{c]d},$$

from which we can rewrite (24) in terms of $\bar{\nabla}$

$$R^a{}_{bcd} = \bar{R}^a{}_{bcd} - 2\bar{\nabla}_{[c} L^a{}_{d]b} + 2L^a{}_{r[c} L^r{}_{d]b}. \tag{26}$$

Of course this last equation could have been obtained from (24) by means of the replacements $R^a{}_{bcd} \leftrightarrow \bar{R}^a{}_{bcd}$, $L^a{}_{bc} \to -L^a{}_{bc}$ and $\nabla_a \to \bar{\nabla}_a$.

Example 3. To realize the importance of the bi-conformal connection in future calculations, let us calculate its components for a *conformally separable* pseudo-Riemannian manifold (see Definition 11) given in local coordinates $x \equiv \{x^1, \ldots, x^n\}$ by

$$ds^2 = \Xi_1(x^1, \ldots, x^n) G_{\alpha\beta}(x^\delta)\, dx^\alpha\, dx^\beta + \Xi_2(x^1, \ldots, x^n) G_{AB}(x^C)\, dx^A\, dx^B. \tag{27}$$

Here Greek indexes range from 1 to p and uppercase Latin indexes from $p + 1$ to n so the metric tensors $G_{\alpha\beta}$ and G_{AB} are of rank p and $n - p$ respectively. The tensors $G^{\alpha\beta}$ and G^{AB} are defined in the obvious way and they are used to raise Greek and uppercase Latin indexes respectively. The non-zero Christoffel symbols for this metric are

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2\Xi_1} G^{\alpha\rho}(\partial_\beta(\Xi_1 G_{\alpha\rho}) + \partial_\gamma(\Xi_1 G_{\rho\beta}) - \partial_\rho(\Xi_1 G_{\beta\gamma})),$$

$$\Gamma^A{}_{BC} = \frac{1}{2\Xi_1} G^{AD}(\partial_B(\Xi_1 G_{CD}) + \partial_C(\Xi_1 G_{DB}) - \partial_D(\Xi_1 G_{BC})),$$

$$\Gamma^\alpha{}_{\beta A} = \frac{1}{2\Xi_1} \delta^\alpha{}_\beta \partial_A \Xi_1, \qquad \Gamma^A{}_{B\alpha} = \frac{1}{2\Xi_2} \delta^A{}_B \partial_\alpha \Xi_2,$$

from which the only nonvanishing components of M_{abc}, E_a, W_a are

$$M_{\alpha AB} = \partial_\alpha(\Xi_2 G_{AB}), \qquad M_{A\alpha\beta} = -\partial_A(\Xi_1 G_{\alpha\beta}),$$
$$E_A = -\,\partial_A \log|\det(\Xi_1 G_{\alpha\beta})|, \qquad W_\alpha = -\partial_\alpha \log|\det(\Xi_2 G_{AB})|. \tag{28}$$

Therefore we get for the components of the bi-conformal connection

$$\bar{\Gamma}^{\alpha}{}_{\beta\phi} = \frac{1}{2\Xi_1}(\delta^{\alpha}{}_{\beta}\partial_{\phi}\Xi_1 + \delta^{\alpha}{}_{\phi}\partial_{\beta}\Xi_1 - G^{\alpha\rho}G_{\beta\phi}\partial_{\rho}\Xi_1) + \Gamma^{\alpha}{}_{\beta\phi}(G),$$
$$\bar{\Gamma}^{A}{}_{BC} = \frac{1}{2\Xi_2}(\delta^{A}{}_{B}\partial_{C}\Xi_2 + \delta^{A}{}_{C}\partial_{B}\Xi_2 - G^{AR}G_{BC}\partial_{R}\Xi_2) + \Gamma^{A}{}_{BC}(G),$$
$$\bar{\Gamma}^{\alpha}{}_{\beta C} = \bar{\Gamma}^{A}{}_{B\phi} = 0, \tag{29}$$

where $\Gamma^{\alpha}{}_{\beta\phi}(G)$ and $\Gamma^{A}{}_{BC}(G)$ are the Christoffel symbols of the metrics $G_{\alpha\beta}$ and G_{AB}, respectively. From the above formulae we deduce that the bi-conformal connection is fully adapted to a conformally separable pseudo-Riemannian manifold because its components clearly split in two parts being each of them the Christoffel symbols of the metrics $G_{\alpha\beta}$, G_{AB} plus terms involving the derivatives of the factors Ξ_1 and Ξ_2. We will take advantage of this property in Section 5 where we will find an invariant local characterization of conformally separable pseudo-Riemannian manifolds.

We calculate next the covariant derivative with respect to the bi-conformal connection of a number of tensors.

Proposition 4. *The following identities hold true*

$$\bar{\nabla}_a P_{bc} = \nabla_a P_{bc} - \frac{1}{p}E_a P_{bc} - \frac{1}{2p}(E_b P_{ac} + E_c P_{ab}) - \frac{1}{2}(P_{cp}M^{p}{}_{ab} + P_{bp}M^{p}{}_{ac}), \tag{30}$$

$$2\bar{\nabla}_a P^{b}{}_{c} = 2\nabla_a P^{b}{}_{c} + P^{bq}P^{r}{}_{c}M_{qra} - \Pi^{bq}P^{r}{}_{c}M_{qra} - P^{b}{}_{q}M^{q}{}_{ac} + \frac{1}{n-p}W_c\Pi^{b}_{a} - \frac{1}{p}E_c P^{b}{}_{a}, \tag{31}$$

$$\bar{\nabla}_a P^{bc} = \nabla_a P^{bc} + \frac{1}{p}E_a P^{bc} + \frac{1}{2(n-p)}(W^c\Pi^{b}{}_{a} + W^b\Pi^{c}{}_{a}) - \frac{1}{2}(M^{b}{}_{ar}P^{rc} + M^{c}{}_{ar}P^{rb}), \tag{32}$$

and all the identities formed with the replacements $P_{ab} \to \Pi_{ab}$, $p \to n-p$.

Proof. All these identities are proven by means of (25) and the use of properties (13). □

Using the above properties we can get more interesting identities to be used later on.

$$\bar{\nabla}_a P^{ab} = \bar{\nabla}_a \Pi^{ab} = \bar{\nabla}_a P^{a}{}_{b} = \bar{\nabla}_a \Pi^{a}_{b} = 0, \tag{33}$$

$$P^{bc}\bar{\nabla}_a P_{bc} = -E_a, \qquad P_{bc}\bar{\nabla}_a P^{bc} = E_a, \tag{34}$$

$$\Pi^{bc}\bar{\nabla}_a \Pi_{bc} = -W_a, \qquad \Pi_{bc}\bar{\nabla}_a \Pi^{bc} = W_a, \tag{35}$$

$$P^{d}{}_{r}\bar{\nabla}_b \Pi^{r}{}_{d} = \Pi^{d}{}_{r}\bar{\nabla}_b P^{r}{}_{d} = P^{dr}\bar{\nabla}_d P_{rb} = \Pi^{dr}\bar{\nabla}_d \Pi_{rb} = 0. \tag{36}$$

Note that index raising and lowering do not commute with $\bar{\nabla}$ so we must be very careful when we raise or lower indexes in tensor expressions involving $\bar{\nabla}$.

4. Normal form and dimension of maximal Lie algebras of bi-conformal fields

We turn now our attention to the calculation of the full normal form coming from the differential conditions (5). A detailed explanation of the general procedure and relevance of this calculation for a general symmetry can be found in [8,10] (see also [24] for the calculation in the cases of the most studied symmetries in General Relativity such as isometries and conformal motions). Before starting the calculation and for the sake of completeness let us give a very brief sketch of the whole procedure. We must differentiate the condition (5) a number of times in such a way that we get enough equations to isolate the derivatives of certain variables (system variables) in terms of themselves (this is achieved typically by means of the resolution of a linear system of equations). The so obtained derivatives give rise to the normal form associated to our symmetry. Along the differentiating process one may obtain equations whose linear combinations no longer contain derivatives of the system variables (constraints). Examples of such constraints in our case are the differential conditions themselves and (12) (actually these are the only constraints as we will show in Section 4.2).

It is possible to meet cases in which the normal form cannot be achieved. This means that one cannot obtain enough equations to isolate all the derivatives obtained through the derivation process. The main implication of this is that the Lie algebra of vector fields fulfilling the starting differential condition is infinite dimensional as opposed to the case in which there is such a normal form. Therefore the calculation of the normal form allows us to tell apart the cases with an infinite dimensional Lie algebra of vector fields from those representing finite dimensional Lie algebras. In the latter case we can even go further and determine the highest dimension of these Lie algebras as the total number of system variables minus the number of linearly independent constraints.

We start out our calculation with the substitution of (22) into (A.1) which yields

$$\begin{aligned}&\bar{\nabla}_b\Psi^a{}_c+\xi^d\bar{R}^a{}_{cdb}=\tfrac{1}{2}(\bar{\phi}_bP^a{}_c+\bar{\phi}_cP^a{}_b-\bar{\phi}^aP_{cb}+\bar{\chi}_b\Pi^a{}_c+\bar{\chi}_c\Pi^a{}_b-\bar{\chi}^a\Pi_{cb}),\\&\Psi_ca\equiv\bar{\nabla}_c\xi^a.\end{aligned}\tag{37}$$

Next we replace in (A.3) the Lie derivatives of the bi-conformal connection by their expressions given by (22) getting

$$\begin{aligned}\pounds_{\vec{\xi}}\,\bar{R}^d{}_{cab}=&\;\bar{\nabla}_{[a}\bar{\phi}_{b]}P^d{}_c+P^d{}_{[b}\bar{\nabla}_{a]}\bar{\phi}_c-P_{c[b}\bar{\nabla}_{a]}\bar{\phi}^d+\bar{\nabla}_{[a}\bar{\chi}_{b]}\Pi^d{}_c+\Pi^d{}_{[b}\bar{\nabla}_{a]}\bar{\chi}_c\\&-\Pi_{c[b}\bar{\nabla}_{a]}\bar{\chi}^d+\bar{\phi}_{[b}\bar{\nabla}_{a]}P^d{}_c+\bar{\phi}_c\bar{\nabla}_{[a}P^d{}_{b]}-\bar{\phi}^d\bar{\nabla}_{[a}P_{b]c}+\bar{\chi}_{[b}\bar{\nabla}_{a]}\Pi^d{}_c\\&+\bar{\chi}_c\bar{\nabla}_{[a}\Pi^d{}_{b]}-\bar{\chi}^d\bar{\nabla}_{[a}\Pi_{b]c}.\end{aligned}\tag{38}$$

The game is now to isolate from this expression $\bar{\nabla}_a\bar{\phi}_b$ and $\bar{\nabla}_a\bar{\chi}_b$ (these rank-2 tensors are not symmetric in general). The forthcoming calculations split in two groups which are dual under the interchange $P_{ab}\Leftrightarrow\Pi_{ab}$, $p\Leftrightarrow n-p$ (only the calculations of the first group are shown). Multiplying (38) by $P^a{}_r$ we obtain

$$\begin{aligned}\pounds_{\vec{\xi}}\,(P^d{}_r\overline{R}^r{}_{cab}) = {} & P^d{}_c\overline{\nabla}_{[a}\overline{\phi}_{b]} + P^d{}_{[b}\overline{\nabla}_{a]}\overline{\phi}_c - P^d{}_r P_{c[b}\overline{\nabla}_{a]}\overline{\phi}^r - P^d{}_r \Pi_{c[b}\overline{\nabla}_{a]}\overline{\chi}^r \\ & + P^d{}_r\overline{\phi}_{[b}\overline{\nabla}_{a]}P^r{}_c + \overline{\phi}_c P^d{}_r\overline{\nabla}_{[a}P^r{}_{b]} - \overline{\phi}^d\overline{\nabla}_{[a}P_{b]c} \\ & + P^d{}_r\overline{\chi}_{[b}\overline{\nabla}_{a]}\Pi^r{}_c + \overline{\chi}_c P^d{}_r\overline{\nabla}_{[a}\Pi^r{}_{b]}.\end{aligned} \tag{39}$$

Contraction of the indexes d-c in the above expression yields

$$\overline{\nabla}_a\overline{\phi}_b = \overline{\nabla}_b\overline{\phi}_a + \frac{2}{p}\,\pounds_{\vec{\xi}}\,(P^d{}_r\overline{R}^r{}_{dab}), \tag{40}$$

while the contraction of indexes d-a and use of identities (33)-(36) entails

$$2\,\pounds_{\vec{\xi}}\,(P^d{}_r\overline{R}^r{}_{cdb}) = P^d{}_c\overline{\nabla}_d\overline{\phi}_b + P^d{}_b\overline{\nabla}_d\overline{\phi}_c - \overline{\phi}^r\overline{\nabla}_r P_{bc} - \overline{\nabla}_a\overline{\phi}^a P_{bc} - p\overline{\nabla}_b\overline{\phi}_c. \tag{41}$$

Eqs. (40) and (41) can now be combined in a single expression which is

$$\begin{aligned} & 2\,\pounds_{\vec{\xi}}\left[P^d{}_r\overline{R}^r{}_{cdb} - \frac{1}{p}(P^d{}_c P^r{}_q\overline{R}^q{}_{rdb} + P^d{}_b P^r{}_q\overline{R}^q{}_{rdc} - P^r{}_q\overline{R}^q{}_{rbc})\right] \\ & \quad = (2-p)\overline{\nabla}_b\overline{\phi}_c - \overline{\nabla}_a\overline{\phi}^a P_{bc} - \overline{\phi}_d(\overline{\nabla}_b P^d{}_c + \overline{\nabla}_c P^d{}_b + P^{rd}\overline{\nabla}_r P_{bc}).\end{aligned} \tag{42}$$

Multiplying here with P^{cb} leads, after a little bit of algebra, to

$$\overline{\nabla}_a\overline{\phi}^a = \frac{1}{1-p}(\pounds_{\vec{\xi}}\,\overline{R}^0 + \phi\overline{R}^0), \quad p \neq 1, \qquad \overline{R}^0 = P^d{}_r\overline{R}^r{}_{cdb}P^{cb}, \tag{43}$$

On the other hand (42) can be further simplified if we take into account the identity

$$\begin{aligned}\overline{\nabla}_b P^d{}_c + \overline{\nabla}_c P^d{}_b - P^{rd}\overline{\nabla}_r P_{bc} = {} & \Pi^{rd}\nabla_r P_{bc} + \frac{1}{2}(\Pi^{dq}\Pi^r{}_c M_{qrb} + \Pi^{dq}\Pi^r{}_b M_{qrc}) \\ & + \frac{1}{2(n-p)}(W_c\Pi^d{}_b + W_b\Pi^d{}_c),\end{aligned}$$

which is easily derived by writing all the covariant derivatives with respect to the bi-conformal connection of the projectors in terms of ordinary covariant derivatives (Proposition 4). So plugging (43) into (42) we get

$$(2-p)\overline{\nabla}_b\overline{\phi}_c = \pounds_{\vec{\xi}}\,L^0{}_{bc} + 2\overline{\phi}^r\overline{\nabla}_r P_{bc}, \tag{44}$$

where

$$L^0{}_{bc} \equiv 2\left[P^d{}_r\overline{R}^r{}_{cdb} - \frac{1}{p}(P^d{}_c P^r{}_q\overline{R}^q{}_{rdb} + P^d{}_b P^r{}_q\overline{R}^q{}_{rdc} - P^r{}_q\overline{R}^q{}_{rbc})\right] + \frac{\overline{R}^0}{1-p}P_{bc}. \tag{45}$$

The duals of (44) and (45) are

$$(2-n+p)\overline{\nabla}_b\overline{\chi}_c = \pounds_{\vec{\xi}}\,L^1{}_{bc} + 2\overline{\chi}^r\overline{\nabla}_r\Pi_{bc}, \tag{46}$$

and

$$L^1{}_{bc} \equiv 2\left[\Pi^d{}_r\overline{R}^r{}_{cdb} - \frac{1}{n-p}(\Pi^d{}_c\Pi^r{}_q\overline{R}^q{}_{rdb} + \Pi^d{}_b\Pi^r{}_q\overline{R}^q{}_{rdc} - \Pi^r{}_q\overline{R}^q{}_{rbc})\right] + \frac{\overline{R}^1}{1-n+p}\Pi_{bc},\ \overline{R}^1 \equiv \Pi^d{}_r\overline{R}^r{}_{cdb}\Pi^{cb}. \tag{47}$$

To complete the normal form we need now the derivatives of ϕ^*_a and χ^*_a which are obtained through the differentiation of (12) (identity (A.2) must be used to get these derivatives)

$$-p\overline{\nabla}_b\phi^*_a = £_{\vec{\xi}}(\overline{\nabla}_b E_a) + \tfrac{1}{2}(\overline{\chi}_b E_a + \overline{\chi}_a E_b - (\overline{\chi}^r E_r)\Pi_{ab}), \tag{48}$$

$$(p-n)\overline{\nabla}_b\chi^*_a = £_{\vec{\xi}}(\overline{\nabla}_b W_a) + \tfrac{1}{2}(\overline{\phi}_b W_a + \overline{\phi}_a W_b - (\overline{\phi}^r W_r)P_{ab}) \tag{49}$$

4.1. Normal form of the differential conditions

The above calculations give us the sought normal form for the differential conditions (5) being these gathered in the following set of equations

$$\begin{aligned}
&\overline{\nabla}_a\phi = \overline{\phi}_a + \phi^*_a, \qquad \overline{\nabla}_a\chi = \overline{\chi}_a + \chi^*_a, && \text{(a)}\\
&\overline{\nabla}_b\phi^*_a = \tfrac{-1}{p}\left[£_{\vec{\xi}}(\overline{\nabla}_b E_a) + \tfrac{1}{2}(\overline{\chi}_b E_a + \overline{\chi}_a E_b - (\overline{\chi}^r E_r)\Pi_{ab})\right], && \text{(b)}\\
&\overline{\nabla}_b\chi^*_a = \tfrac{1}{p-n}\left[£_{\vec{\xi}}(\overline{\nabla}_b W_a) + \tfrac{1}{2}(\overline{\phi}_b W_a + \overline{\phi}_a W_b - (\overline{\phi}^r W_r)P_{ab})\right], && \text{(c)}\\
&\overline{\nabla}_b\overline{\phi}_c = \tfrac{1}{2-p}[£_{\vec{\xi}} L^0_{bc} + 2\overline{\phi}^r\overline{\nabla}_r P_{bc}], && \text{(d)}\\
&\overline{\nabla}_b\overline{\chi}_c = \tfrac{1}{2-n+p}[£_{\vec{\xi}} L^1_{bc} + 2\overline{\chi}^r\overline{\nabla}_r\Pi_{bc}], && \text{(e)}\\
&\overline{\nabla}_b\xi^a = \Psi_b a, && \text{(f)}\\
&\overline{\nabla}_b\Psi_c a = \tfrac{1}{2}(\overline{\phi}_b P^a{}_c + \overline{\phi}_c P^a{}_b - \overline{\phi}^a P_{cb} + \overline{\chi}_b\Pi^a{}_c + \overline{\chi}_c\Pi^a{}_b - \overline{\chi}^a\Pi_{cb}) - \xi^d\overline{R}^a{}_{cdb}. && \text{(g)}
\end{aligned} \tag{50}$$

A first glance at these equations reveals us that this normal form does not always exist. To be precise if either $p = 2$ or $p = n - 2$ the derivatives $\overline{\nabla}_a\overline{\phi}_b$ and $\overline{\nabla}_a\overline{\chi}_b$ cannot be isolated and the system cannot be "closed" (in fact these derivatives cannot be isolated even if we perform further derivatives of any of the above equations). We must also remember at this point that the tensors L^0_{ab} and L^1_{ab} are well-defined unless $p = 1$ and $p = n - 1$ respectively (see Eqs. (45) and (47)). Therefore we have proven the following theorem (compare to Proposition 6.2 of [10])

Theorem 5. *The only cases in which the Lie algebra $\mathcal{G}(S)$ can be infinite dimensional occur if and only if $p = 1$, $p = 2$, $p = n - 1$, $p = n - 2$.*

The result of this theorem is intuitively clear if we realize that bi-conformal vector fields are somehow conformal motions for the projectors P_{ab} and Π_{ab}. Therefore if any of them projects onto one or two dimensional vector spaces the associated Lie algebras may turn out to be infinite dimensional as we have just found.

Eqs. (50) are no longer a normal form system if some of the derivatives involved vanish. This happens for instance if part of the gauge functions are constants or their second covariant derivatives with respect to the bi-conformal connection are zero. In all this work we will assume that these derivatives are not zero in an open neighbourhood of a point leaving the study of any other cases for a forthcoming publication.

The normal form can be used to establish the minimum conditions under which bi-conformal vector fields are smooth vector fields.

Proposition 6. *Let $\vec{\boldsymbol{\xi}}$ be a bi-conformal vector field at least C^2 in a neighbourhood $\mathcal{U}_x$ of a point x belonging to a manifold V with a C^∞ metric tensor. If ϕ, χ are at least C^2 on $\mathcal{U}_x$ then $\vec{\boldsymbol{\xi}} \in C^\infty(\mathcal{U}_x)$.*

Proof. To prove this result it is enough to show that the covariant derivatives of $\vec{\boldsymbol{\xi}}$ with respect to the bi-conformal connection exist at any order. The first and second derivatives of $\vec{\boldsymbol{\xi}}$ are Eqs. (50)-f and (50)-g and higher derivatives are calculated from this last equation. When we derive (50)-g we need $\overline{\nabla}_b\overline{\phi}_c$ and $\overline{\nabla}_b\overline{\chi}_c$ which exist on $\mathcal{U}_x$ as ϕ, $\chi \in C^2(\mathcal{U}_x)$ and these are obtained through Eq. (50)-d and (50)-e in which only derivatives of $\vec{\boldsymbol{\xi}}$, $\overline{\phi}_a$ and $\overline{\chi}_a$ of order less or equal than one appear. This makes clear that no other equation of (50) but the ones mentioned so far are involved in the calculation of the derivatives of $\vec{\boldsymbol{\xi}}$ and so we can obtain them in as high order as we wish. □

Related to this is the following result (the technique used in the proof has been employed in [12] for other symmetries in the framework of General Relativity).

Proposition 7. *Under the hypotheses of previous proposition if a bi-conformal vector field $\vec{\boldsymbol{\xi}}$ is such that $\xi^a|_x = 0$, $\overline{\nabla}_b\xi^a|_x = 0$, $\overline{\nabla}_c\overline{\nabla}_b\xi^a|_x = 0$ then $\vec{\boldsymbol{\xi}} \equiv 0$ in a neighbourhood of x.*

Proof. Evaluation of the last equation of (50) at x entails

$$(\overline{\phi}_b P^a{}_c + \overline{\phi}_c P^a{}_b - \overline{\phi}^a P_{cb} + \overline{\chi}_b \Pi^a{}_c + \overline{\chi}_c \Pi^a{}_b - \overline{\chi}^a \Pi_{cb})|_x = 0,$$

from which, projecting down with P^{ab} and Π^{ab}, we deduce that $\overline{\phi}_a|_x = \overline{\chi}_a|_x = 0$. Now let $\gamma(t)$ be a smooth curve on V such that $\gamma(0) = x$ with $\gamma(t)$ lying in a coordinate neighbourhood of x for all t in the interval $(-\epsilon, \epsilon)$. If we denote by $\dot{\gamma}^a(t)$ the tangent vector to this curve we may define the derivatives

$$\frac{\overline{D}\overline{\phi}_a}{\mathrm{d}t} \equiv \dot{\gamma}^r\overline{\nabla}_r\overline{\phi}_a, \qquad \frac{\overline{D}\overline{\chi}_a}{\mathrm{d}t} \equiv \dot{\gamma}^r\overline{\nabla}_r\overline{\chi}_a, \qquad \frac{\overline{D}\xi^a}{\mathrm{d}t} \equiv \dot{\gamma}^r\overline{\nabla}_r\xi^a, \qquad \frac{\overline{D}\Psi_c a}{\mathrm{d}t} \equiv \dot{\gamma}^r\overline{\nabla}_r\Psi_c a,$$

where all quantities are evaluated on $\gamma(t)$. Contracting (50)-d–(50)-g with $\dot{\gamma}^b$ we can transform these equations into a first order ODE system in the variables $\overline{\phi}_a(\gamma(t))$, $\overline{\chi}_a(\gamma(t))$, $\xi^a(\gamma(t))$ and $\Psi_a b(\gamma(t))$ with $\overline{D}/\mathrm{d}t$ as derivation. From the above all these variables vanish at $t = 0$ so according to the standard theorem of uniqueness for ODE systems the variables are identically zero along the curve $\gamma(t)$ (and in particular $\vec{\boldsymbol{\xi}}|_{\gamma(t)} = 0$). As $\gamma(t)$ was chosen arbitrarily we conclude that $\vec{\boldsymbol{\xi}} \equiv 0$ in a whole neighbourhood of x. □

Remark 8. If the manifold is connected (and hence path connected) then the curve $\gamma(t)$ can be chosen joining any pair of points of the manifold and then the vector field $\vec{\boldsymbol{\xi}}$ is zero everywhere and not just in a single neighbourhood of a point.

From the calculations performed in this proof and in the proof of Proposition 6 we deduce as a simple corollary that $\overline{\nabla}^{(m)}\xi^a|_x \equiv 0$ for all $m \in \mathbb{N}$ if it holds for $m = 1, 2$.

4.2. Constraints

From the above calculation of the normal form, the system variables are read off at once. These are the variables appearing under derivation in the left hand side of (50). However, they are not algebraically independent because in the calculation process of (50) some of the equations involved do not contain derivatives of the variables at all (system constraints). The most evident case of these constraints are the differential conditions (5) themselves. If we review the whole procedure followed to get (50) we deduce that the other set of constraints between the system variables is (12) so (50) must be complemented with

$$\begin{aligned} &\text{(I)} \quad \pounds_{\vec{\xi}}\, P_{ab} = \phi P_{ab}, \qquad \pounds_{\vec{\xi}}\, \Pi_{ab} = \chi \Pi_{ab}, \\ &\text{(II)} \quad \pounds_{\vec{\xi}}\, E_a = -p\phi^*_a, \qquad \pounds_{\vec{\xi}}\, W_a = -(n-p)\chi^*_a \end{aligned} \tag{51}$$

In order to clarify that these two sets of equations are truly the constraints associated with (50) we must show that they arise as a linear combination of some of the higher covariant derivatives of (5) employed to get the normal form. An equation equivalent to the first derivative of (5) is (11) but only its projections by P^{ab} and Π^{ab} (Eq. 12) really matter to work out the normal form and these are (51)-II. Eq. (37) is also obtained from (11) and it is part of the normal system and not a constraint. As for the other derivatives they do not give rise to any more equations with no derivatives of the system variables so the above equations are the only constraints we must care about.

A first application of all the above calculations comes in the following result, already proven in [10] using a normal form system written in terms of different variables.

Theorem 9. *If the Lie algebra $\mathcal{G}(S)$ is finite dimensional then its dimension is bounded from above by $N = p(p+1)/2 + (n-p)(n-p+1)/2$.*

Proof. To prove this theorem we must state what the upper bound to the maximum number of *integration constants* for the system (50) is. As is very well known from the theory of normal systems of PDE's (see e.g. [8]) such number is the number of system variables minus the number of linearly independent constraints. The following table summarizes these numbers for our system.

We have written explicitly the system variables and the total number for each of them. The constraints are also indicated together with how many linearly independent equations each constraint amounts to. This last part is not evident as opposed to the counting for the system variables so the rest of the proof is devoted to show that the numbers given in Table 1 for the constraint equations are indeed correct.

Table 1
Calculation of the highest dimension of $\mathcal{G}(S)$

System variables					Constraints	
ϕ, χ	$\phi^*{}_a, \bar{\phi}_a$	$\chi^*{}_a, \bar{\chi}_a$	ξ^a	$\Psi_a b$	Eq. (51)-I	Eq. (51)-II
2	n	n	n	n^2	$n(n+1)/2 + p(n-p)$	n

Eq. (51)-I. First of all, we expand the Lie derivatives of these equations

$$\xi^c \bar{\nabla}_c P_{ab} + \Psi_p c(\delta^p{}_a P_{bc} + \delta^p{}_b P_{ac}) = \phi P_{ab},$$
$$\xi^c \bar{\nabla}_c \Pi_{ab} + \Psi_p c(\delta^p{}_a \Pi_{bc} + \delta^p{}_b \Pi_{ac}) = \chi \Pi_{ab}, \tag{52}$$

where the standard definition of the Lie derivative of a tensor P_{ab} has been applied

$$\pounds_{\vec{\xi}}\, P_{ab} \equiv \xi^c \bar{\nabla}_c P_{ab} + \bar{\nabla}_a \xi^c P_{cb} + \bar{\nabla}_b \xi^c P_{ac}, \tag{53}$$

(observe that the general formula of the Lie derivative with respect to a vector in terms of its covariant derivatives still holds under a symmetric connection). We define new indexes A, B, B' in such a way that

$$P_A \equiv P_{ab}, \qquad \Psi_B \equiv \bar{\nabla}_p \xi^q, \qquad \xi^{B'} = \xi^c,$$

so capital indexes group together certain combinations of small indexes (explicitly $A = \{a, b\}$, $B = \{p, q\}$ and $B' = c$). The ranges of the new indexes are $A = 1, \ldots, n(n+1)/2$, $B = 1, \ldots n^2$, $B' = 1, \ldots n$. Using these new labels we can write in matrix notation the homogeneous system posed by these constraints (we only concentrate in the first of (52))

$$\begin{pmatrix} M_A{}^B & (\bar{\nabla} P)_{B'A} & P_A \end{pmatrix} \begin{pmatrix} \Psi_B \\ \xi^{B'} \\ -\phi \end{pmatrix} = 0, \tag{54}$$

(A = row index, B, B' = column indices) where the explicit expressions of the matrices read

$$(\bar{\nabla} P)_{B'A} = \bar{\nabla}_c P_{ab}, \qquad M_A{}^B = \delta^p{}_a P_{bq} + \delta^p{}_b P_{aq}.$$

The number of linearly independent equations is just the rank of the matrix system of (54). In principle the rank of this matrix will depend on the projector P_{ab} and its covariant derivative, meaning this that it will depend on the geometry of the manifold. However, since we are interested in spaces with a maximum number of bi-conformal vector fields it is enough to find the least rank of the above matrix for all possible projectors P_{ab}. We start first studying the rank of $M_A B$ whose nonvanishing components occur in the following cases (no summation over the repeated indexes)

$$p = a,\ q = b \Rightarrow \begin{cases} M^A{}_A = \delta^a{}_a P_{bb}, & a \neq b, \\ M^A{}_A = 2P_{aa}, & a = b. \end{cases},$$

$$p = b,\ q = a \Rightarrow M^{Q'}{}_A = \delta^b{}_b P_{aa}, \quad a \neq b,$$

where we have assumed that we are working in the common (orthonormal) basis of eigenvectors of $P^a{}_b$ and $\Pi^a{}_b$ so

$$P^a{}_b = \mathrm{diag}(\overbrace{1\ldots1}^{p}\,0\ldots0), \qquad \Pi^a{}_b = \mathrm{diag}(0\ldots0\,\overbrace{1\ldots1}^{n-p}).$$

Hence we only need to count how many components of the type $M_A{}^A$ are different from zero because by construction these elements give rise to linearly independent rows of the matrix $M_A{}^B$ (the elements $M_A{}^{Q'}$ are in the same row of the matrix $M_A{}^B$ and they do not increase its rank because $Q' > A$). The sought number can be obtained from the following diagram gathering into blocks the A indexes of the rows containing non-zero elements (we express each index in terms of tensor indexes following the notation $A = (a, b)$)

$$\begin{array}{l|l}
\mathrm{Block}\,1 = \{\overbrace{(1,1)\,(1,2)\,\ldots\,(1,p)}^{p}\} & \mathrm{Block}\,p+1 = \{\overbrace{(p+1,1)\,\ldots(p+1,p)}^{p}\} \\
\mathrm{Block}\,2 = \{\overbrace{(2,2)\,(2,3)\,\ldots\,(2,p)}^{p-1}\} & \mathrm{Block}\,p+2 = \{\overbrace{(p+2,1)\,\ldots\,(p+2,p)}^{p}\} \\
\mathrm{Block}\,3 = \{\overbrace{(3,3)\,(3,4)\,\ldots\,(3,p)}^{p-2}\} & \mathrm{Block}\,p+3 = \{\overbrace{(p+3,1)\,\ldots\,(p+3,p)}^{p}\} \\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
\mathrm{Block}\,p = \{\overbrace{(p,p)}^{1}\} & \mathrm{Block}\,p+n = \{\overbrace{(n,1)\,\ldots\,(n,p)}^{p}\},
\end{array}$$

from which the rank of $M_A{}^B$ is

$$1 + \cdots + p + p(n-p) = \tfrac{1}{2}p(p+1) + p(n-p).$$

Note that this rank only depends on algebraic properties of the projector P_{ab} and not on its actual form at some concrete space. Addition of the matrices $(\bar{\nabla} P)_{B'A}$ and P_A only increase the rank of the matrix of the homogeneous system (54) and so we do not need to take them into account. The total number of constraints posed by (51)-I is then the rank of $M_A{}^B$ plus the rank of the matrix $N_A{}^B$ constructed replacing P_{ab} by Π_{ab}

$$\begin{aligned}\mathrm{rank}(M) + \mathrm{rank}(N) &= \tfrac{1}{2}p(p+1) + p(n-p) + \tfrac{1}{2}(n-p)(n-p+1) + p(n-p) \\ &= \tfrac{1}{2}n(n+1) + p(n-p).\end{aligned}$$

Eq. (51)-II. In order to perform the analysis of these constraints it is enough to realize that the 1-forms ϕ^*_a and χ^*_a appearing in the right hand side of each equation are invariant under the projectors $\Pi^a{}_b$ and $P^a{}_b$ respectively. Therefore the first equation of (51)-II contains at least $n-p$ linearly independent equations and the second one p being n the total sum of them.

The upper bound N is then

$$\begin{aligned}N &= 2 + n + n + n + n^2 - (n + \tfrac{1}{2}n(n+1) + p(n-p)) \\ &= \tfrac{1}{2}(p+1)(p+2) + \tfrac{1}{2}(n-p+1)(n-p+2). \qquad \square\end{aligned}$$

This proof does not guarantee the existence of a Lie algebra $\mathcal{G}(S)$ in which the dimension N is attained. However, it is not difficult to find explicit examples of pseudo-Riemannian manifold possessing N bi-conformal vector fields.

Proposition 10. *The number N is the maximum dimension of $\mathcal{G}(S)$ if $p, n-p \notin \{1, 2\}$ being this dimension attained for any pseudo-Riemannian manifold whose line element is in local coordinates $\{x^a\}$, $a = 1, \ldots, n$*

$$\mathrm{d}s^2 = \phi_1^2(x^a)\eta^0_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \phi_2^2(x^a)\eta^1_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B, \tag{55}$$

where $x^\alpha = \{x^1, \ldots, x^p\}$, $x^A = \{x^{p+1}, \ldots, x^n\}$ are sets of coordinates and $\boldsymbol{\eta}^0$, $\boldsymbol{\eta}^1$ flat metrics of the appropriate signatures depending only on the coordinates $\{x^\alpha\}$ and $\{x^A\}$ respectively.

Proof. This result is proposition 6.1 of [10]. □

Spaces of previous proposition are called *bi-conformally flat*. As stated in Definition 12 they are a particular case of *conformally separable spaces* and we may ask if they are the only pseudo-Riemannian manifolds admitting N independent bi-conformal vector fields. The answer to this and other questions such as their geometric characterization can be settled by calculating the complete integrability conditions of (50). Remarkably this has been already done but the whole procedure relies on hefty algebraic manipulations so we have preferred to present these results in a subsequent paper. The interested reader can find the full details of these calculations in [11].

5. Local geometric characterization of conformally separable pseudo-Riemannian manifolds

In this section we will show how the bi-conformal connection can be used to derive an invariant geometric characterization of conformally separable pseudo-Riemannian manifolds. To begin with we define in precise terms what a conformally separable pseudo-Riemannian manifold is.

Definition 11. The pseudo-Riemannian manifold (V, g) is said to be conformally separable at the point $q \in V$ if there exists a local coordinate chart $x \equiv \{x^1, \ldots, x^n\}$ based at q in which the metric tensor takes the form

$$\mathrm{g}_{ab}(x) = \begin{cases} \Xi_1(x)G_{\alpha\beta}(x^\gamma), & 1 \le \alpha, \beta, \gamma \le p \\ \Xi_2(x)G_{AB}(x^C), & p+1 \le A, B, C \le n \\ 0 & \text{otherwise.} \end{cases} \tag{56}$$

where Ξ_1, Ξ_2 are C^2 functions on the open set defining the coordinate chart. (V, g) is conformally separable if it is so at every point $p \in V$. Any of the metric tensors $\Xi_1 G_{\alpha\beta}$, $\Xi_2 G_{AB}$ shall be called a leaf metric.

Henceforth all our results will deal with conformally separable pseudo-Riemannian manifolds at a point. From now on when working with conformally separable spaces written in the form of (56) we adopt the convention that Greek letters label indexes associated to one of the leaf metrics whereas uppercase Latin characters are used for the other one.

Conformally separable pseudo-Riemannian manifolds are also known as *double twisted products*. They comprise a number of particular cases which have received wide attention in the literature under different nomenclatures. A summary of them is presented next.

Definition 12. Let $x \equiv \{x^a\}$, $a = 1 \dots n$ be the local coordinate system introduced in Definition 11. A conformally separable manifold can then be classified in terms of the form that the functions Ξ_1, Ξ_2 take in these coordinates as

(1) decomposable or reducible: $\mathrm{d}s^2 = G_{\alpha\beta}(x^\epsilon)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + G_{AB}(x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B$,
(2) semi-decomposable, semi-reducible or warped product: $\mathrm{d}s^2 = G_{\alpha\beta}(x^\epsilon)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \Xi(x^\epsilon)G_{AB}(x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B$, $\Xi(x^\epsilon)$ warping factor,
(3) generalized decomposable or double warped: $\mathrm{d}s^2 = \Xi_1(x^C)G_{\alpha\beta}(x^\epsilon)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \Xi_2(x^\epsilon)G_{AB}(x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B$, $\Xi_1(x^C)$, $\Xi_2(x^\epsilon)$ warping factors,
(4) twisted product: $\mathrm{d}s^2 = G_{\alpha\beta}(x^\epsilon)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \Xi_2(x^a)G_{AB}(x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B$
(5) conformally reducible: $\mathrm{d}s^2 = \Xi(x^a)(G_{\alpha\beta}(x^\epsilon)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + G_{AB}(x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B)$,
(6) bi-conformally flat: $\mathrm{d}s^2 = \Xi_1(x^a)\eta_{\alpha\beta}(x^\epsilon)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \Xi_2(x^a)\eta_{AB}(x^C)\,\mathrm{d}x^A\,\mathrm{d}x^B$, $\eta_{\alpha\beta}$, η_{AB} flat metrics of dimension p and $n-p$, respectively.

The coordinate system of Definitions 11 and 12 is fully adapted to the decomposition of the metric tensor but in general we cannot expect this to be the case. Therefore it would be desirable to have a result characterizing conformally separable pseudo-Riemannian manifolds or any of the cases presented in Definition 12 in a coordinate-free way. Next we prove an intrinsic *local* characterization valid for a general conformally separable pseudo-Riemannian manifold which will enable us to derive characterizations for most of the particular cases described in Definition 12 in a simple way. A lemma is needed first.

Lemma 13. *The following assertion is true*

$$T_{abc} = 0 \Longleftrightarrow \bar{\nabla}_a P_{bc} = -\frac{1}{p} E_a P_{bc}, \qquad \bar{\nabla}_a \Pi_{bc} = -\frac{1}{n-p} W_a \Pi_{bc}. \tag{57}$$

Proof. First of all, it is convenient to rewrite the condition $T_{abc} = 0$ in an appropriate form. From (16) we have

$$M_{abc} = \frac{1}{p} E_a P_{bc} - \frac{1}{n-p} W_a \Pi_{bc}. \tag{58}$$

Using the definition $M_{abc} = \nabla_b P_{ac} + \nabla_c P_{ab} - \nabla_a P_{bc}$ we can isolate $\nabla_b P_{ac}$ getting

$$\nabla_b P_{ac} = \frac{1}{2p}(E_a P_{bc} + E_c P_{ab}) - \frac{1}{2(n-p)}(W_a \Pi_{bc} + W_c \Pi_{ba}). \tag{59}$$

Each (58) and (59) are equivalent to $T_{abc} = 0$. Next we show the equivalence of $T_{abc} = 0$ to the conditions (57). Expanding $\bar{\nabla}_a P_{bc}$ by means of (30) and use of (57) yields

$$\nabla_b P_{ac} = \frac{1}{2p}(E_a P_{bc} + E_c P_{ab}) + \frac{1}{2}(P_{cp} M^p{}_{ba} + P_{ap} M^p{}_{bc}) \tag{60}$$

$$\nabla_b \Pi_{ac} = \frac{1}{2(n-p)}(W_a \Pi_{bc} + W_c \Pi_{ab}) - \frac{1}{2}(\Pi_{cp} M^p{}_{ba} + \Pi_{ap} M^p{}_{bc}), \tag{61}$$

which are equivalent to (write $\nabla_a P_{bc}$, $\nabla_a \Pi_{bc}$ in terms of M_{abc})

$$P_{cp} M^p{}_{ab} = -\frac{1}{n-p} W_c \Pi_{ab}, \qquad \Pi_{cp} M^p{}_{ab} = \frac{1}{p} E_c P_{ab}, \tag{62}$$

whose addition leads to $T_{abc} = 0$. Conversely, suppose that $T_{abc} = 0$. Then inserting (58) and (59) into (30) gives us the condition on $\overline{\nabla}_a P_{bc}$ at once. The calculation for $\overline{\nabla}_a \Pi_{bc}$ is similar using the identities written in terms of Π_{ab}. □

Theorem 14. *A pseudo-Riemannian manifold* (V, g) *is conformally separable at the point* $p \in V$ *if and only if there exists an orthogonal projector* P_{ab} *such that the tensor* T_{abc} *formed with* P_{ab} *and its complementary* $\Pi_{ab} = \mathrm{g}_{ab} - P_{ab}$ *is zero identically in a neighbourhood of p. In such case* P_{ab} *and* Π_{ab} *are the leaf metrics of the separation.*

Proof. To show that the condition of the theorem is necessary we simply choose the local coordinates around p in which the metric tensor takes the form of (56) and calculate the tensor T_{abc} as in Example 3. Use of (28) readily implies that $T_{abc} = 0$. To prove that the condition is also sufficient choose an orthonormal co-basis $\{\overline{\boldsymbol{\theta}}^1, \ldots, \overline{\boldsymbol{\theta}}^n\}$ adapted to P_{ab} and Π_{ab}, that is to say, (we use index-free notation and index label splitting as in Definition 11)

$$\boldsymbol{P} = \sum_{\alpha=1}^{p} \epsilon_\alpha \overline{\boldsymbol{\theta}}^\alpha \otimes \overline{\boldsymbol{\theta}}^\alpha, \qquad \boldsymbol{\Pi} = \sum_{A=p+1}^{n} \epsilon_A \overline{\boldsymbol{\theta}}^A \otimes \overline{\boldsymbol{\theta}}^A,$$

where $\epsilon_\alpha,\ \epsilon_A = \pm 1$ (the exact value for each index α, A will depend on the signature of g_{ab}). Now since

$$\overline{\nabla}_c \overline{\boldsymbol{\theta}}^\alpha = -\overline{\gamma}^\alpha{}_{bc} \overline{\boldsymbol{\theta}}^b = -\overline{\gamma}^\alpha{}_{\beta c} \overline{\boldsymbol{\theta}}^\beta - \overline{\gamma}^\alpha{}_{Bc} \overline{\boldsymbol{\theta}}^B,$$

we have

$$\overline{\nabla}_c \boldsymbol{P} = -\sum_{\alpha=1}^{p} \epsilon_\alpha [\overline{\gamma}^\alpha{}_{\beta c}(\overline{\boldsymbol{\theta}}^\beta \otimes \overline{\boldsymbol{\theta}}^\alpha + \overline{\boldsymbol{\theta}}^\alpha \otimes \overline{\boldsymbol{\theta}}^\beta) + \overline{\gamma}^\alpha{}_{Bc}(\overline{\boldsymbol{\theta}}^B \otimes \overline{\boldsymbol{\theta}}^\alpha + \overline{\boldsymbol{\theta}}^\alpha \otimes \overline{\boldsymbol{\theta}}^B)],$$

which by (57) entails $\overline{\gamma}^\alpha{}_{Bc} = 0$. Similarly condition (57) on $\boldsymbol{\Pi}$ shows that $\overline{\gamma}^B{}_{\alpha c} = 0$. These two conditions upon the connection coefficients imply by means of Frobenius theorem that the distributions spanned by $\{\overline{\boldsymbol{\theta}}^1, \ldots, \overline{\boldsymbol{\theta}}^p\}$ and $\{\overline{\boldsymbol{\theta}}^{p+1}, \ldots, \overline{\boldsymbol{\theta}}^n\}$ are both integrable. Therefore in a local coordinate system $\{x^1, \ldots, x^n\}$ around p adapted to the manifolds generated by these distributions (i.e. in these coordinates the manifolds are given by the conditions $x^\alpha = c^\alpha$, $x^A = c^A$) the metric tensor takes the form

$$\mathrm{d}s^2 = \mathrm{g}_{\alpha\beta}(x^a)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta + \mathrm{g}_{AB}(x^a)\,\mathrm{d}x^A\,\mathrm{d}x^B,$$

and the tensors P_{ab} and Π_{ab} look like

$$P_{ab} = \mathrm{g}_{\alpha\beta}\delta^{\alpha}{}_{a}\delta^{\beta}{}_{b}, \qquad \Pi_{ab} = \mathrm{g}_{AB}\delta^{A}{}_{a}\delta^{B}{}_{b},$$

so the non-zero components of the Christoffel symbols are

$$\Gamma^{\alpha}{}_{\beta\gamma} = \tfrac{1}{2}\mathrm{g}^{\alpha\rho}(\partial_\beta \mathrm{g}_{\gamma\rho} + \partial_\gamma \mathrm{g}_{\rho\beta} - \partial_\rho \mathrm{g}_{\beta\gamma}), \qquad \Gamma^{\alpha}{}_{\beta A} = \tfrac{1}{2}\mathrm{g}^{\alpha\rho}\partial_A \mathrm{g}_{\beta\rho},$$
$$\Gamma^{\alpha}{}_{BA} = -\tfrac{1}{2}\mathrm{g}^{\alpha\rho}\partial_\rho \mathrm{g}_{BA}, \qquad \Gamma^{A}{}_{B\alpha} = \tfrac{1}{2}\mathrm{g}^{AD}\partial_\alpha \mathrm{g}_{BD}, \qquad \Gamma^{A}{}_{\alpha\beta} = -\tfrac{1}{2}\mathrm{g}^{AD}\partial_D \mathrm{g}_{\beta\alpha},$$
$$\Gamma^{A}{}_{BC} = \tfrac{1}{2}\mathrm{g}^{AD}(\partial_B \mathrm{g}_{CD} + \partial_C \mathrm{g}_{DB} - \partial_D \mathrm{g}_{BC}),$$

where

$$\mathrm{g}^{\alpha\rho}\mathrm{g}_{\rho\beta} = \delta^{\alpha}{}_{\rho}, \qquad \mathrm{g}^{AD}\mathrm{g}_{DB} = \delta^{A}{}_{B}.$$

The only nonvanishing components of M_{abc}, E_a, W_a are thus

$$M_{\alpha AB} = \partial_\alpha \mathrm{g}_{AB}, \qquad M_{A\alpha\beta} = -\partial_A \mathrm{g}_{\alpha\beta}, \qquad E_A = -\partial_A \log|\det(\mathrm{g}_{\alpha\beta})|,$$
$$W_\alpha = -\partial_\alpha \log|\det(\mathrm{g}_{AB})|, \tag{63}$$

from which we deduce that those of T_{abc} are

$$T_{\alpha AB} = \partial_\alpha \mathrm{g}_{AB} + \frac{1}{n-p}\mathrm{g}_{AB}W_\alpha, \qquad T_{A\alpha\beta} = \partial_A \mathrm{g}_{\alpha\beta} + \frac{1}{p}\mathrm{g}_{\alpha\beta}E_A, \tag{64}$$

Thus we are left with the couple of equations (64) equalled to zero. The general solution of the resulting PDE system is

$$\mathrm{g}_{\alpha\beta} = G_{\alpha\beta}(x^\delta)\,\mathrm{e}^{\Lambda_1(x^a)}, \qquad \mathrm{g}_{AB} = G_{AB}(x^D)\,\mathrm{e}^{\Lambda_2(x^a)},$$

where $G_{\alpha\beta}$, G_{AB}, Λ_1, Λ_2 are arbitrary functions of their respective arguments with no restrictions other than $\det(G_{\alpha\beta}) \neq 0$, $\det(G_{AB}) \neq 0$. Comparing these expression with (56) the result follows. □

Remark 15. A global characterization of conformally separable pseudo-Riemannian manifolds was first given in [23] and is this: a pseudo-Riemannian manifold is conformally separable iff there exist two orthogonal families of foliations by totally umbilical hypersurfaces. The family of first fundamental forms of each hypersurface gives rise to the leaf metrics of the decomposition of g_{ab} in the obvious way (this result was re-derived in [16]).

Theorem 14 clearly states the geometric relevance of T_{abc} as a tool to characterize conformally separable pseudo-Riemannian manifolds. In fact the condition $T_{abc} = 0$ can be re-written in terms of the factors Ξ_1 and Ξ_2 introduced in the definition of a conformally separable metric. To that end we use the equivalent condition (59) and replace the 1-forms E_a, W_a by their expressions given in (63) which can be written as

$$E_a = -p\Pi_a{}^r\partial_r \log|\Xi_1|, \qquad W_a = (p-n)P_a{}^r\partial_r \log|\Xi_2|, \tag{65}$$

whence

$$\nabla_b P_{ac} = P_{bc}u_a + P_{ab}u_c - P_a{}^r u_r g_{bc} - P_c{}^r u_r g_{ab}, \tag{66}$$

where

$$u_a = \frac{E_a}{2p} + \frac{W_a}{2(n-p)}.$$

It is not difficult now to characterize intrinsically almost all the subcases presented in Definition 12.

Theorem 16. *Under the hypotheses of Theorem 14 a pseudo-Riemannian manifold* (V, g) *is locally*

(1) *decomposable or reducible if and only if* $E_a = W_a = 0$,
(2) *a warped product if and only if* $E_a = 0$ *and* W_a *is an exact* 1*-form,*
(3) *a double warped product if and only if both* E_a, W_a *are exact* 1*-form,*
(4) *a twisted product if and only if* $E_a = 0$,
(5) *conformally reducible if and only if* u_a *is exact.*

In all cases the conditions are understood to hold only in a neighbourhood of a point p.

Proof. To show that the above conditions are necessary we only have to apply formula (65) case by case and take into account that under the conditions of Theorem 14

$$P^a{}_b = \begin{cases} \delta^\alpha{}_\beta, & a = \alpha, b = \beta, \\ 0 & \text{otherwise.} \end{cases}, \qquad \Pi^a{}_b = \begin{cases} \delta^A{}_B, & a = A, b = B \\ 0 & \text{otherwise.} \end{cases},$$

where the local coordinates of (56) have been set around p.

- (V, g) decomposable or reducible $\Rightarrow \Xi_1 = \Xi_2 = 1 \Rightarrow E_a = W_a = 0$.
- (V, g) warped product $\Rightarrow E_a = 0$, $W_a = (p-n)\partial_a \log|\Xi_2|$.
- (V, g) double warped $\Rightarrow E_a = -p\partial_a \log|\Xi_1|$, $W_a = (p-n)\partial_a \log|\Xi_2|$.
- (V, g) twisted product $\Rightarrow \Xi_1 = 1 \Rightarrow E_a = 0$.
- (V, g) conformally reducible $\Rightarrow \Xi_1 = \Xi_2 = \Xi \Rightarrow u_a = -\partial_a \log|\Xi|^{1/2}$.

The sufficiency follows from simple algebraic manipulations involving the conditions of each case and the relations

$$E_A = -p\frac{\partial_A \Xi_1}{\Xi_1}, \qquad W_A = -(n-p)\frac{\partial_\alpha \Xi_2}{\Xi_2},$$

coming from (28) which holds due to the property $T_{abc} = 0$.

- $E_a = W_a = 0 \Rightarrow \partial_A \Xi_1 = \partial_\alpha \Xi_2 = 0 \Rightarrow \Xi_1 = \Xi_1(x^\alpha),\ \Xi_2 = \Xi_2(x^A) \Rightarrow (V, \mathrm{g})$ is decomposable or reducible.
- $E_a = 0$ and W_a exact $\Rightarrow \Xi_1 = \Xi_1(x^\alpha)$ and for some scalar function Φ we have

$$-(n-p)\frac{\partial_\alpha \Xi_2}{\Xi_2} = \partial_\alpha \Phi, \qquad 0 = \partial_A \Phi,$$

  which entails $\Xi_2 = \Xi_2(x^\alpha) \Rightarrow (V, \mathrm{g})$ is a warped product.
- E_a and W_a are both exact. A calculation similar to the previous point leads to $\Xi_1 = \Xi_1(x^A)$, $\Xi_2 = \Xi_2(x^\alpha) \Rightarrow (V, \mathrm{g})$ is double warped.
- $E_a = 0 \Rightarrow \Xi_1 = \Xi_1(x^\alpha) \Rightarrow (V, \mathrm{g})$ is a twisted product.
- If u_a is exact then for some scalar function Φ we have

$$-\tfrac{1}{2}\partial_\alpha \log|\Xi_2| = \partial_\alpha \Phi, \qquad -\tfrac{1}{2}\partial_A \log|\Xi_1| = \partial_A \Phi$$

  which implies that either (V, g) is a double twisted product (and in particular conformally reducible with $\Xi = \Xi_1(x^A)\Xi_2(x^\alpha)$) or $|\Xi_1| = |\Xi_2| \Rightarrow (V, \mathrm{g})$ is conformally reducible. □

Local characterizations of some of the cases presented in Theorem 16 are already known and have been rediscovered several times by different procedures. For instance the reducibility condition is clearly equivalent to $\nabla_a P_{bc} = \nabla_a \Pi_{bc} = 0$ which was proven in [15] in the context of General Relativity ([25] also proves this result in Riemannian geometry). This result is known in Riemannian geometry as *de Rham decomposition theorem* and it was formulated by de Rham in both local and global terms [6] (the global version was formulated as early as in 1924 [2]). A local characterization of Riemannian warped products is sometimes attributed to Hiepko [13] but in [14] such characterization is already present. Alternative local characterizations to those of Theorem 16 of double warped products and certain conformally reducible manifolds were found in [4,3] in the framework of General Relativity and general conformally reducible Riemannian manifolds were locally characterized in [1]. There are also *global characterizations* of the cases discussed in Theorem 16 (see [9] for a summary of them).

In any case our method is more general and simpler than the procedures followed so far and it covers virtually all possible types of conformally reducible pseudo-Riemannian manifolds being all of them presented in a single general result (Theorem 16). This makes of the bi-conformal connection an important tool in the study of conformally separable pseudo-Riemannian manifolds and we believe that it could play a key role in the research of these manifolds because it is the natural connection which keeps the decomposition of the metric tensor in the two leaf metrics defined by the projectors P_{ab} and Π_{ab}. An explicit example of this key role is the local characterization of conformally separable pseudo-Riemannian manifolds with a conformally flat leaf metric, not covered by Theorem 16. These cases have never been tackled before and in the companion paper following this publication we show how the use of the bi-conformal connection allows us to find a local characterization in the same terms as Theorem 16. The impatient reader may consult all the details in [11] but just to give a first glimpse we will say that this characterization comes in through the vanishing of a four rank tensor involving the curvature tensor of the bi-conformal connection (in a sense it resembles the local characterization of conformally flat pseudo-Riemannian manifolds by means of the vanishing of the Weyl tensor). More examples showing explicitly the usefulness of our techniques are supplied in next section.

6. Examples

Example 17. As our first example we consider the four dimensional pseudo-Riemannian manifold with metric given by

$$\mathrm{d}s^2 = (\Psi^2 \sin^2\theta - \alpha^2)\,\mathrm{d}t^2 + 2\Psi^2 \sin^2\theta\,\mathrm{d}\phi\,\mathrm{d}t + B^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) + \Phi^2 \sin^2\theta\,\mathrm{d}\phi^2,$$

where the coordinate ranges are $-\infty < t < \infty,\ 0 < r < \infty,\ 0 < \theta < \pi,\ 0 < \theta < 2\pi$ and the functions Ψ, α, B and Φ only depend on the coordinates r, θ. We will try to find out the conditions under which the metric is conformally separable with the hypersurfaces $t =$ const as one of the leaves. A simple calculation shows that the projector $P^a{}_b$ projecting vectors onto the distribution generated by the above hypersurfaces is (now and henceforth all the components omitted in an explicit tensor representation are understood to be zero)

$$P^r{}_r = P^\theta{}_\theta = P^\phi{}_\phi = 1, \qquad P^\phi{}_t = \frac{\Psi^2}{\Phi^2},$$

which entails

$$P_{tt} = \frac{\Psi^4}{\Phi^2}\sin^2\theta, \qquad P_{rr} = B^2, \qquad P_{\theta\theta} = r^2B^2,$$
$$P_{\phi\phi} = \Phi^2 \sin^2\theta, \qquad P_{t\phi} = \Psi^2 \sin^2\theta.$$

From here we can calculate the components of the tensor T_{abc} and set them equal to zero. After doing this we find the following independent conditions (letter subscripts mean partial derivatives)

$$-\Psi\Phi_r + \Psi_r\Phi = 0, \qquad -\Psi\Phi_\theta + \Psi_\theta\Phi = 0,$$

which are fulfilled if and only if

$$|\Phi| = |\Psi|$$

Under these conditions the metric takes the form

$$\mathrm{d}s^2 = \Psi^2 \sin^2\theta(\mathrm{d}t + \mathrm{d}\phi)^2 - \alpha^2\,\mathrm{d}t^2 + B^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2).$$

This metric is not written in the form of (56) and so it is not evident that it is conformally separable with the hypersurfaces $t =$ const as the leaves. This is so because the remaining coordinates r, θ, ϕ are not adapted to the separation and so a coordinate change would be necessary to bring the above metric into the form (56). An advantage of our technique is that we do not need to find this coordinate change and only by prescribing one of the leaves of the separation have we been able to determine easily that our pseudo-Riemannian manifold is conformally separable. In this particular example we can even go further and calculate the 1-forms E_a and W_a. In this way we obtain that $E_a = 0$ whereas W_a is closed (locally exact) so Theorem 16 says that this pseudo-Riemannian manifold is locally a warped product.

Example 18. In the foregoing results we have only concentrated on conformally separable pseudo-Riemannian manifolds but nothing was said about manifolds with *conformal slices* and not conformally separable. To illustrate this case let us consider the four dimensional pseudo-Riemannian manifold given in local coordinates $\{x^1, x^2, x^3, x^4\}$ by

$$\begin{aligned} \mathrm{d}s^2 = {} & \Phi(x)[\Xi_1(x^1, x^2, x^3)(\mathrm{d}x^1)^2 + \Xi_2(x^1, x^2, x^3)(\mathrm{d}x^2)^2 + \Xi_3(x^1, x^2, x^3)(\mathrm{d}x^3)^2] \\ & + 2\sum_{i=1}^{3} \beta_i(x)\mathrm{d}x^i\,\mathrm{d}x^4 + \Psi(x)(\mathrm{d}x^4)^2, \end{aligned} \tag{67}$$

where $x = \{x^1, x^2, x^3, x^4\}$ and $\Phi(x)$, $\beta_i(x)$, $\Psi(x)$, $\{\Xi_i(x^1, x^2, x^3)\}_{i=1,2,3}$ are functions at least C^2 in an open domain. The above line element is the most general four dimensional metric admitting a local foliation by three dimensional conformal hypersurfaces (here these are given by the condition $x^4 = \text{const}$) because according to a classical result any three dimensional metric tensor can be written as the bracket term multiplying $\Phi(x)$ in Eq. (67).

The non-zero components of the orthogonal projector $P^a{}_b$ associated to the foliation $x^4 = \text{const}$ (see previous example) are

$$P^1{}_1 = P^2{}_2 = P^3{}_3 = 1, \qquad P^i{}_4 = \frac{\beta_i(x)}{\Phi(x)\Xi_i(x^1, x^2, x^3)}, \quad i = 1, 2, 3, \tag{68}$$

from which we easily get

$$P_{11} = P_{22} = P_{33} = \Phi(x), \qquad P_{i4} = \beta_i(x), \quad i = 1, 2, 3,$$

$$P_{44} = \sum_{i=1}^{3} \frac{\beta_i^2(x)}{\Phi(x)\Xi_i(x^1, x^2, x^3)}.$$

Using this we can check the condition $T_{abc} = 0$ and find out what is obtained. This is a rather long calculation which is easily performed with any of the computer algebra systems available today (the system used here was GRTensorII). The result is that the tensor T_{abc} does not vanish in this case although a calculation using (68) shows the important property

$$P^r{}_a P^s{}_b P^q{}_c T_{rsq} = 0. \tag{69}$$

Theorem 19. *A necessary condition that a four dimensional pseudo-Riemannian manifold can be foliated by conformal hypersurfaces with associated orthogonal projector P_{ab} is* Eq. (69).

This result suggests that it may well be possible to generalize conditions of Theorem 14 to metrics of arbitrary dimension which are not conformally separable replacing these conditions by (69). The true extent of this assertion is under current research.

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Appendix A. Basic identities involving the Lie derivative

In this appendix we recall some properties of the Lie derivative needed in the main text. Despite their basic character, they are hardly presented in basic Differential Geometry textbooks and the author is only aware of [24,18] as the only references in which they are studied.

Proposition A.1. *For any symmetric connection* $\overline{\nabla}$ *defined in a differentiable manifold* V, *any vector field* $\vec{\boldsymbol{\xi}}$ *at least* C^2 *and a tensor field* $T^{a_1\ldots a_p}{}_{b_1\ldots b_q} \in T^p{}_q(V)$ *we have the following identities*

$$\pounds_{\vec{\xi}}\, \overline{\gamma}^a{}_{bc} = \overline{\nabla}_b \overline{\nabla}_c \xi^a + \xi^d \overline{R}^a{}_{cdb}, \tag{A.1}$$

$$\begin{aligned} \overline{\nabla}_c\, \pounds_{\vec{\xi}}\, T^{a_1\ldots a_s}{}_{b_1\ldots b_q} - \pounds_{\vec{\xi}}\, \overline{\nabla}_c T^{a_1\ldots a_s}{}_{b_1\ldots b_q} &= -\sum_{j=1}^{s} (\pounds_{\vec{\xi}}\, \overline{\gamma}^{a_j}{}_{cr}) T^{\ldots a_{j-1} r a_{j+1}\ldots}{}_{b_1\ldots b_q} \\ &\quad + \sum_{j=1}^{q} (\pounds_{\vec{\xi}}\, \overline{\gamma}^r_{cb_j}) T^{a_1\ldots a_s}{}_{\ldots b_{j-1} r b_{j+1}\ldots}, \end{aligned} \tag{A.2}$$

$$\pounds_{\vec{\xi}}\, \overline{R}^d{}_{cab} = \overline{\nabla}_a (\pounds_{\vec{\xi}}\, \overline{\gamma}^d{}_{bc}) - \overline{\nabla}_b (\pounds_{\vec{\xi}}\, \overline{\gamma}^d{}_{ac}), \tag{A.3}$$

where $\overline{\gamma}^a{}_{bc}$ are the components of the connection $\overline{\nabla}$ and $\overline{R}^a{}_{bcd}$ its curvature (these identities are calculated under the convention (4) for the curvature tensor). Furthermore if a metric tensor g_{ab} is set in V and ∇ is now the metric connection associated to it then

$$\pounds_{\vec{\xi}}\, \gamma^a{}_{bc} = \tfrac{1}{2} g^{ae} [\nabla_b (\pounds_{\vec{\xi}}\, g_{ce}) + \nabla_c (\pounds_{\vec{\xi}}\, g_{be}) - \nabla_e (\pounds_{\vec{\xi}}\, g_{bc})]. \tag{A.4}$$

Remark A.2. The Lie derivative of the connection is a tensor even though $\gamma^a{}_{bc}$ is not. To see this we denote by $\{\Phi_s\}$ the one-parameter group of local diffeomorphisms generated by the vector field $\vec{\boldsymbol{\xi}}$ and by $(\Phi^*_s \gamma)^a{}_{bc}$ the transformed of the connection under Φ_s which in the local coordinates $x = \{x^1, \ldots, x^n\}$ is calculated by means of the formula

$$(\Phi^*{}_s \Gamma)^a{}_{bc}(x) = \frac{\partial \Phi^a{}_{-s}}{\partial x^r} \frac{\partial \Phi^z{}_s}{\partial x^b} \frac{\partial \Phi^q{}_s}{\partial x^c} \Gamma^r{}_{zq}(\Phi_s(x)) + \frac{\partial \Phi^a{}_{-s}}{\partial x^r} \frac{\partial}{\partial x^b} \left(\frac{\partial \Phi^r{}_s}{\partial x^c} \right).$$

Hence neither $(\Phi_s^*\gamma)^a{}_{bc}$ nor $\gamma^a{}_{bc}$ are tensors but the difference $(\Phi_s^*\gamma)^a{}_{bc} - \gamma^a{}_{bc}$ is a tensor and this implies that

$$\lim_{s\to 0} \frac{(\Phi_s^*\gamma)^a{}_{bc} - \gamma^a{}_{bc}}{s}$$

is also a tensor which is the Lie derivative of the connection.

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